PMM U.S.S.R.,Vol.49,No.6.pp.756-763,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

SOFT ELASTIC SHELLS UNDERGOING LARGE DEFORMATIONS*

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Soft shells made of elastomers and undergoing large deformations under load are studied. The inverse design problem, non-linear under large deformations, is solved. The results obtained are illustrated on a two-parameter shell of revolution fabricated from a two-constant material. The problems of coupling the biaxial and uniaxial zones of the shell and of designing the composite shell are clarified. Amongst the papers dealing with the theory of soft shells and, generally, under small deformations, /1–7/ merit attention.

1. The equations of the equilibrium of the zero-moment theory of thin shells have the form /8, 9/

$$\frac{\partial \sqrt{a^\circ}\, T^{\gamma j}}{\partial \alpha^\gamma} + \Gamma_{\gamma\beta}^j \sqrt{a^\circ}\, T^{\gamma\beta} + \sqrt{a}\, q^j = 0 \tag{1.1}$$

$$b_{\gamma\beta} \sqrt{a^\circ}\, T^{\gamma\beta} + \sqrt{a}\, q_n = 0 \quad (j, \gamma, \beta = 1,2)$$

Here and henceforth the repeated Greek indices denote summation from 1 to 2.
For an incompressible material we have

$$T^{ij} = 2h^\circ \left[\frac{\partial \Phi^\circ}{\partial A} a^{\circ ij} + \left(B \frac{\partial \Phi^\circ}{\partial B} - B^{-1} \frac{\partial \Phi^\circ}{\partial \lambda_\xi^2} \right) a^{ij} \right] \tag{1.2}$$

$$\lambda_\xi = (a/a^\circ)^{-1/2} = B^{-1/2}, \quad A = a^{\circ\alpha\beta} a_{\alpha\beta} \quad (a^\circ = |a_{ij}^\circ|,\ a = |a_{ij}|) \tag{1.3}$$

(λ_ξ is the extension factor of the transverse fibre and $\Phi^\circ = \Phi|_{\xi=0}$ is the value of the elastic potential on the middle surface of the shell).

The practice of the analysis of thin-walled technical resin items has shown that the deformation of incompressible elastomers is well described by the three-constant elastic potential

$$\Phi^\circ = \mu n^{-2} [(1 + \beta)(\lambda_1^n + \lambda_2^n + \lambda_\xi^n - 3) + (1 - \beta)(\lambda_1^{-n} + \lambda_2^{-n} + \lambda_\xi^{-n} - 3)] \tag{1.4}$$

Here λ_1, λ_2 are the principal extension factors of the middle surface. We also have

$$A = \lambda_1^2 + \lambda_2^2, \quad B = \lambda_1^2 \lambda_2^2, \quad \lambda_\xi = \lambda_1^{-1} \lambda_2^{-1} \tag{1.5}$$

$$\lambda_{1,2} = {}^1/_2 [(A + 2B^{1/2})^{1/2} \pm (A - 2B^{1/2})^{1/2}]$$

For a three-constnat elastic potential the elastic law assumes the form

$$\frac{T^{ij}}{\mu h^\circ n^{-1}} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^2 - \lambda_2^2} [1 + \beta + (1 - \beta) \lambda_1^{-n} \lambda_2^{-n}] a^{\circ ij} + \tag{1.6}$$

$$(\lambda_1^2 - \lambda_2^2)^{-1} \{(1 + \beta)[\lambda_1^2 \lambda_2^n - \lambda_1^n \lambda_2^2 - (\lambda_1^2 - \lambda_2^2) \lambda_1^{-n} \lambda_2^{-n}] +$$

$$(1 - \beta)[\lambda_1^{-n} \lambda_2^2 - \lambda_1^2 \lambda_2^{-n} + (\lambda_1^2 - \lambda_2^2) \lambda_1^n \lambda_2^n]\} a^{ij}$$

On the principal axes of deformation $a^{012} = a^{12} = 0$, $a^{ii} = a^{0ii} \lambda_i^{-2}$ and

$$T^{11} = \mu h^\circ n^{-1} [1 + \beta + (1 - \beta) \lambda_2^n] \lambda_1^{-2} (\lambda_1^n - \lambda_1^{-n} \lambda_2^{-n}) / a_{11}^\circ \tag{1.7}$$

$$T^{22} = \mu h^\circ n^{-1} [1 + \beta - (1 - \beta) \lambda_1^n] \lambda_2^{-2} (\lambda_2^n - \lambda_1^{-n} \lambda_2^{-n}) / a_{22}^\circ, \quad T^{12} = 0$$

In particular, for a neo-Hookean material ($\beta = 1$, $n = 2$)

$$T^{ij} = \mu h^\circ (a^{\circ ij} - B^{-1} a^{ij}) \tag{1.8}$$

and on the principal axes we have

$$T^{11} = \mu h^\circ (1 - \lambda_1^{-n} \lambda_2^{-2}) / a_{11}^\circ, \quad T^{22} = \mu h^\circ (1 - \lambda_1^{-2} \lambda_2^{-n}) / a_{22}^\circ, \quad T^{12} = 0$$

The shell thickness as before and after deformation (h° and h) are connected, in the case of an incompressible material, by the relation

$$h = \lambda_\xi h^\circ = (a/a^\circ)^{-1/2} h^\circ = \lambda_1^{-1} \lambda_2^{-1} h_\circ \tag{1.9}$$

The physical components of the inherent stress tensor referred to the metric of the deformed middle surface are found from the formulas

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$$\sigma_{(ij)} = \sqrt{\frac{a_{ii}a_{jj}}{B}}\,\frac{T^{ij}}{h}$$

If $\mathbf{r} = x_1\mathbf{g}_1 + x_2\mathbf{g}_2 + x_3\mathbf{g}_3$ represents the radius vector of a point on the middle surface in the rectangular Cartesian coordinate space, then (Fig.1)

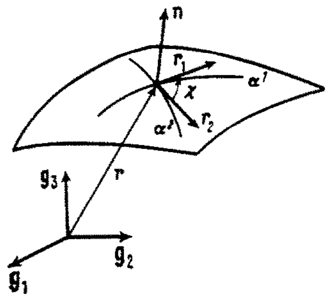

Fig.1

$$\mathbf{r}_i = \frac{\partial x_1}{\partial \alpha^i}\mathbf{g}_1 + \frac{\partial x_2}{\partial \alpha^i}\mathbf{g}_2 + \frac{\partial x_3}{\partial \alpha^i}\mathbf{g}_3 \tag{1.10}$$

$$a_{ij} = \frac{\partial x_1}{\partial \alpha^i}\frac{\partial x_1}{\partial \alpha^j} + \frac{\partial x_2}{\partial \alpha^i}\frac{\partial x_2}{\partial \alpha^j} + \frac{\partial x_3}{\partial \alpha^i}\frac{\partial x_3}{\partial \alpha^j}$$

$$a = a_{11}a_{22} - a_{12}{}^2 = \sin^2\chi = \mu^2 = \mu_{13}{}^2 + \mu_{23}{}^2 + \mu_{33}{}^2$$

$$a^{11} = a_{22}/a, \quad a^{22} = a_{11}/a, \quad a^{12} = -a_{12}/a$$

$$\mu\mathbf{n} = \mu_{13}\mathbf{g}_1 + \mu_{23}\mathbf{g}_2 + \mu_{33}\mathbf{g}_3$$

$$\sin\chi\, b_{ij} = \mu_{13}\frac{\partial^2 x_1}{\partial\alpha^i\partial\alpha^j} + \mu_{23}\frac{\partial^2 x_2}{\partial\alpha^i\partial\alpha^j} + \mu_{33}\frac{\partial^2 x_3}{\partial\alpha^i\partial\alpha^j}$$

$$\sqrt{a_{11}a_{22}}\,\mu_{13} = \frac{\partial(x_2, x_3)}{\partial(\alpha^1, \alpha^2)}, \quad \sqrt{a_{11}a_{22}}\,\mu_{23} = \frac{\partial(x_3, x_1)}{\partial(\alpha^1, \alpha^2)}$$

$$\sqrt{a_{11}a_{22}}\,\mu_{33} = \frac{\partial(x_1, x_2)}{\partial(\alpha^1, \alpha^2)}$$

$$\Gamma_{ij}{}^h = {}^1/_2\left(\frac{\partial a_{i\beta}}{\partial\alpha^j} + \frac{\partial a_{j\beta}}{\partial\alpha^i} - \frac{\partial a_{ij}}{\partial\alpha^\beta}\right)a^{\beta h}$$

The corresponding relations for the undeformed middle surface follow from the above formulas by adding the superscript °.

The contravariant components of the stress tensor are connected with its principal values by the relations

$$a_{11}T^{11} = \cos^2\gamma t_1 + \sin^2\gamma t_2, \quad a_{22}T^{22} = \sin^2\gamma t_1 + \cos^2\gamma t_2 \tag{1.11}$$
$$\sqrt{a_{11}a_{22}}\,T^{12} = \sin\gamma\cos\gamma\,(t_1 - t_2)$$

where γ is the angle between the first coordinate line and the first coordinate direction.

2. As we know /1, 2/, the soft shells have very low flexural rigidity and do not, therefore, take up any compressive stresses. A soft shell may have biaxial zones (in which the principal stresses are positive), and uniaxial zones (in which one of the principal stresses is positive and the other is vanishingly small so that it can be assumed equal to zero).

Suppose for example /1/

$$t_1 > 0, \quad t_2 = 0 \tag{2.1}$$

in the uniaxial zones. Taking the orthogonal lines of the principal stresses as the coordinate lines in the uniaxial zones we obtain (for $\gamma = 0$) from (1.11)

$$T^{11} = a_{11}{}^{-1}t_1, \quad T^{12} = T^{22} = 0 \tag{2.2}$$

and the equilibrium Eqs.(1.1) under normal pressure ($q_n = q = \text{const}$, $q^1 = q^2 = 0$) take the form

$$\frac{1}{\sqrt{a_{11}}}\frac{\partial\sqrt{a_{11}}\sqrt{a^{\circ}}\,T^{11}}{\partial\alpha^1} = 0, \quad -\frac{\sqrt{a_{11}}}{a_{22}}\frac{\partial\sqrt{a_{11}}}{\partial\alpha^2}\sqrt{a^{\circ}}\,T^{11} = 0 \tag{2.3}$$

$$-\frac{a_{11}}{R_1}\sqrt{a^{\circ}}\,T^{11} + \sqrt{a_{11}a_{22}}\,q = 0$$

$$\left(\frac{1}{R_1} = -\frac{b_{11}}{a_{11}}, \quad \frac{1}{R_2} = -\frac{b_{22}}{a_{22}}, \quad \frac{1}{R_{12}} = \frac{b_{12}}{\sqrt{a_{11}a_{22}}}\right)$$

The second of these equations yields

$$a_{11} = a_{11}(\alpha^1) \tag{2.4}$$

i.e. the first coordinate line is a geodesic. From the first and third equation we have

$$\sqrt{a^{\circ}}\,T^{11} = q\frac{c_2(\alpha^2)}{\sqrt{a_{11}}}, \quad \frac{1}{R_1} = \frac{\sqrt{a_{22}}}{c_2(\alpha^2)} \tag{2.5}$$

Another three equations yield the Codazz-Gauss /6/ relations which can be written, taking both Eq.(2.4) and the second relation of (2.5) into account, in the form

$$\frac{\partial}{\sqrt{a_{11}}\,\partial\alpha^1}\left(\frac{\sqrt{a_{22}}}{R_2}\right) + \frac{\partial}{\partial\alpha^2}\left(\frac{1}{R_{12}}\right) = \frac{\sqrt{a_{22}}}{c_2(\alpha^2)}\frac{\partial\sqrt{a_{22}}}{\sqrt{a_{11}}\,\partial\alpha^1} \tag{2.6}$$

$$\frac{\partial}{\partial\alpha^2}\left(\frac{\sqrt{a_{22}}}{c_2(\alpha^2)}\right) + \frac{1}{\sqrt{a_{22}}}\frac{\partial}{\sqrt{a_{11}}\,\partial\alpha^1}\left(\frac{a_{22}}{R_{12}}\right) = 0$$

$$\frac{\partial}{\sqrt{a_{11}}\,\partial\alpha^1}\left(\frac{\partial\sqrt{a_{22}}}{\sqrt{a_{11}}\,\partial\alpha^1}\right) = -\sqrt{a_{22}}\left[\frac{\sqrt{a_{22}}}{c_2(\alpha^2)\,R_2} - \frac{1}{R_{12}{}^2}\right]$$

3. One of the fundamental problems for a soft shell is that of design. We require to design (plan) the shell such that it takes the necessary form under the given loads and support conditions. Since the geometry of the shell and the load are both given, the system of three equilibrium equations is linear with respect to the forces sought

$$\sqrt{a^\circ}T^{11},\quad \sqrt{a^\circ}T^{12}=\sqrt{a^\circ}T^{21},\quad \sqrt{a^\circ}T^{22}$$

In the case of an incompressible material we obtain, from (1.2) and (1.9), a system of three (essentially different) non-linear algebraic equations for determining the metric tensor of the undeformed middle surface

$$\frac{\partial\Phi^\circ}{\partial A}a^{\circ ij}+\left(B\frac{\partial\Phi^\circ}{\partial B}-B^{-1}\frac{\partial\Phi^\circ}{\partial\lambda_\xi^2}\right)a^{ij}=\frac{\sqrt{a^\circ}T^{ij}}{2h\sqrt{a}}=f^{ij}\tag{3.1}$$

The above relation, taking Eqs.(1.5) and the obvious relation $a^{\alpha\beta}a_{\alpha\beta}=2$ into account, gives the following invariance relations:

$$A\frac{\partial\Phi^\circ}{\partial A}+2\left(B\frac{\partial\Phi^\circ}{\partial B}-B^{-1}\frac{\partial\Phi^\circ}{\partial\lambda_\xi^2}\right)=a_{\alpha\beta}f^{\alpha\beta}\tag{3.2}$$

$$\left(B-\frac{1}{4}A^2\right)\left(\frac{\partial\Phi^\circ}{\partial A}\right)^2=a\left[f^{11}f^{22}-(f^{12})^2\right]-\frac{1}{4}(a_{\alpha\beta}f^{\alpha\beta})$$

Having found the invariants A and B from the non-linear algebraic ststem given above, we obtain from (3.1) and (1.10)

$$\frac{a^{\circ ij}}{a^{ij}}=\left[\frac{f^{ij}}{a^{ij}}-\left(B\frac{\partial\Phi^\circ}{\partial B}-B^{-1}\frac{\partial\Phi^\circ}{\partial\lambda_\xi^2}\right)\right]\Big/\frac{\partial\Phi^\circ}{\partial A}\tag{3.3}$$

$$a_{11}{}^\circ=aB^{-1}a^{\circ 22},\quad a_{12}{}^\circ=-aB^{-1}a^{\circ 12},\quad a_{22}{}^\circ=aB^{-1}a^{\circ 11}\tag{3.4}$$

In particular, for the neo-Hookean law (1.8) we have

$$B^3-4a\mu^{-2}\left[f^{11}f^{22}-(f^{12})^2\right]B^2-2a_{\alpha\beta}f^{\alpha\beta}\mu^{-1}B-1=0$$

$$a^{\circ ij}=B^{-1}a^{ij}+\frac{\sqrt{a^\circ}T^{ij}}{\mu h\sqrt{a}}$$

4. The formulas obtained can also be used for the uniaxial zone. Thus relations (2.2) and first relation of (2.5) yield

$$f^{11}=\frac{qc_2(\alpha^2)}{2ha_{11}\sqrt{a_{22}}},\quad f^{12}=f^{22}=0\tag{4.1}$$

and, since $a^{12}=0$, it follows from the relations (3.3), (3.4) that $a_{12}{}^\circ=0$. Thus the material coordinate lines are principle (and orthogonal) for the deformation tensor. Moreover, according to (1.10) we have

$$\sqrt{a^\circ}=\sqrt{a_{11}{}^\circ a_{22}{}^\circ},\quad a^{\circ 11}=1/a_{11}{}^\circ,\quad a^{\circ 22}=1/a_{22}{}^\circ\quad (a^{\circ 12}=a_{12}{}^\circ=0)\tag{4.2}$$

$$\sqrt{a}=\sqrt{a^\circ}\lambda_1\lambda_2,\quad a^{11}=1/(\lambda_1{}^2a_{11}{}^\circ),\ a^{22}=1/(\lambda_2{}^2a_{22}{}^\circ)\ (a^{12}=a_{12}=0)$$

and from (4.1) and (3.3) we have

$$\lambda_2{}^2=-\left(B\frac{\partial\Phi^\circ}{\partial B}-B^{-1}\frac{\partial\Phi^\circ}{\partial\lambda_\xi^2}\right)\Big/\frac{\partial\Phi^\circ}{\partial A},\quad \lambda_1{}^2=\lambda_2{}^2+\bar q\Big/\frac{\partial\Phi^\circ}{\partial A}\tag{4.3}$$

$$\bar q=qc_2(\alpha^2)/(2ha_{22})$$

The invariants are found from the set of Eqs.(3.2) whose right-hand sides have the form $\sqrt{a_{22}}\bar q$ and $-(a_{22}\bar q^2)/4$ respectively, while for the neo-Hookean law we have

$$\lambda_2{}^2=B^{-1},\quad \lambda_1{}^2=B^{-1}+\frac{qc_2(\alpha^2)}{\mu h\sqrt{a_{22}}},\quad B^3-\frac{qc_2(\alpha^2)}{\mu h\sqrt{a_{22}}}B-1=0$$

5. In the case when the principal coordinate lines are lines of curvature, /6/, $R_{12}{}^{-1}=0$ and the second Codazz-Gauss relation (2.6) yields $\sqrt{a_{22}}=c_1(\alpha^1)c_2(\alpha^2)$. Substituting the expression obtained into the remaining two equations and carrying out the transformations with the help of relations (2.5), we obtain

$$R_1{}^{-1}=c_1(\alpha^1),\ R_2{}^{-1}={}^1/_2\left[c_1(\alpha^1)-1/(c_1(\alpha^1)R^2)\right]\tag{5.1}$$

$$\sqrt{a^\circ}T^{11}=qc_2(\alpha^2)/\sqrt{a_{11}},\quad a_{11}=a_{11}(\alpha^1)$$

Here $a_{11}(\alpha^1)$, $c_2(\alpha^2)$ are arbitrary functions, R is an arbitrary constant and $c_1(\alpha^1)$ is a solution of the equation

$$d^2c_1/(\sqrt{a_{11}}d\alpha^1)^2 + {}^1/_2\,(c_1{}^3 - c_1R^{-2}) = 0 \tag{5.2}$$

6. Consider the axisymmetric deformation of a shell of revolution (Fig.2). We have

$$\alpha^1 = s^\circ,\ \alpha^2 = \theta^\circ \tag{6.1}$$

By virtue of the assumption that the deformation is axisymmetric, we have

$$x_1{}^\circ = r^\circ(s^\circ)\cos\theta^\circ,\ x_2{}^\circ = r^\circ(s^\circ)\sin\theta^\circ,\ x_3{}^\circ = x_3{}^\circ(s^\circ) \tag{6.2}$$
$$x_1 = r\,(s^\circ)\cos\theta^\circ,\ x_2 = r\,(s^\circ)\sin\theta^\circ,\ x_3 = x_3\,(s^\circ)$$

Let $\lambda_s\,(=\lambda_1)$, $\lambda_\theta\,(=\lambda_2)$ be the principal extension factors along the meridianal and peripheral directions. Fig.3 and its analogue for the deformed configuration imply $(' = \partial/\partial s^\circ)$

$$r^{\circ\prime} = \cos\varphi^\circ,\ x_3{}^{\circ\prime} = -\sin\varphi^\circ,\ r' = \lambda_s\cos\varphi,\ x_3{}' = -\lambda_s\sin\varphi \tag{6.3}$$
$$\lambda_s = ds/ds^\circ,\ \lambda_\theta = r/r^\circ$$

According Eqs.(6.2) and (1.10) we have

$$a_{11}{}^\circ = 1,\ a_{22}{}^\circ = r^{\circ2},\ a^\circ = r^{\circ2},\ a^{\circ11} = 1,\ a^{\circ22} = 1/r^{\circ2} \tag{6.4}$$
$$a_{11} = \lambda_s{}^2,\quad a_{22} = r^{\circ2}\lambda_\theta{}^2,\quad a = (r^\circ\lambda_s\lambda_\theta)^2,\quad a^{11} = \lambda_s{}^{-1},\quad a^{22} = (r^\circ\lambda_\theta)^{-2}$$
$$\Gamma_{11}{}^1 = \lambda_s\lambda_s{}',\quad \Gamma_{12}{}^2 = \lambda_s\ \cos\varphi/(r^\circ\lambda_\theta),\quad \Gamma_{22}{}^1 = -\,r^\circ\lambda_s^{-1}\lambda_\theta\cos\varphi,$$
$$\Gamma_{11}{}^2 = \Gamma_{12}{}^1 = \Gamma_{22}{}^2 = 0$$
$$\frac{1}{R_1} = -\frac{b_{11}}{a_{11}} = \lambda_s^{-1}\varphi',\quad \frac{1}{R_2} = -\frac{b_{22}}{a_{22}} = \sin\varphi\,(r^\circ\lambda_\theta)^{-1},\quad b_{12} = 0$$

Taking the above relations into account, we can write the solution of the equilibrium Eqs.(1.1) in the form

$$\lambda_\theta^{-1}T_s = \mu h\bar\sigma\left[\frac{k+\bar r^2}{\bar r\sin\varphi}\right],\quad \lambda_s^{-1}T_\theta = \mu h\bar\sigma\left[-\frac{k+\bar r^2}{\sin^2\varphi}\frac{d\varphi}{d\bar s} + 2\frac{\bar r}{\sin\varphi}\right] \tag{6.5}$$
$$(T_s = \lambda_sT^{11},\quad T_\theta = r^{\circ2}\lambda_\theta T^{22})$$

where the following dimensionless quantities were used:

$$\bar r = r/R_0,\ \bar s = s/R_0,\ k = P/({}^1/_2qR_0{}^2),\ \bar\sigma = qR_0/(2\mu h) \tag{6.6}$$

Here R_0 is the characteristic linear dimension of the middle surface or the radius of its curvature, q is the uniform pressure and $2\pi P$ is the excess pressure. If the shell is covered from the top and only the normal pressure acts, then $P = 0$ and $k = 0$.

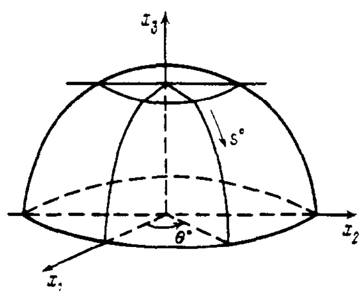

Fig.2

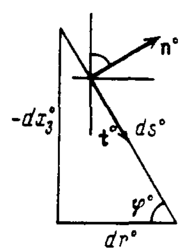

Fig.3

Using the relations (1.7), (6.4) and (6.5) we obtain the following:

$$^1/_2\,[1+\beta+(1-\beta)\,\lambda_\theta{}^n]\,(\lambda_s{}^n - D^{-1}) = f_s \tag{6.7}$$
$$^1/_2\,[1+\beta+(1-\beta)\,\lambda_s{}^n]\,(\lambda_\theta{}^n - D^{-1}) = f_\theta$$
$$\left(f_s = \frac{\lambda_\theta^{-1}T_s}{(2\mu/n)h},\quad f_\theta = \frac{\lambda_s^{-1}T_\theta}{(2\mu/n)h},\quad C = \lambda_s{}^n + \lambda_\theta{}^n,\quad D = \lambda_s{}^n\lambda_\theta{}^n\right)$$

Adding and multiplying the Eqs.(6.7), we obtain a system of equations for determining the invariants C and D. Therefore from (6.7) we obtain

$$\lambda_s{}^n = \frac{1}{2}C + \frac{f_s - f_\theta}{1+\beta+(1-\beta)\,D^{-1}},\ \lambda_\theta{}^n = \frac{1}{2}C - \frac{f_s - f_\theta}{1+\beta+(1-\beta)\,D^{-1}}$$

Now from (6.3) we find, in succession,

$$r^\circ(s) = \lambda_\theta^{-1}(s)\,r\,(s),\quad ds^\circ = \lambda_s^{-1}\,ds \tag{6.8}$$
$$\cos\varphi^\circ = \frac{dr^\circ}{ds^\circ} = \lambda_s\frac{dr^\circ}{ds} = \lambda_s\frac{d\,[\lambda_\theta^{-1}(s)\,r\,(s)]}{ds}$$

$$x_3^c(s) = -\int_{s_1}^{s} \lambda_s^{-1}(s) \sin \varphi^c(s)\, ds, \quad s^c = \int_{s_1}^{s} \lambda_s^{-1}\, ds$$

We recall that expressions (6.5) hold for the biaxial zone, i.e. when the inequalities $(q > 0,\ \bar{\sigma} > 0)$ hold, we have

$$\frac{k + \tau^2}{r \sin \varphi} > 0, \quad -\frac{k + \tau^2}{\sin^2 \varphi} \frac{d\varphi}{ds} + 2 \frac{}{\sin \varphi} > 0$$

When $\beta = 1$, we have according to (1.7)

$$\lambda_\theta^{-1} T_s = 2\mu h \frac{\lambda_s^n - \lambda_s^{-n}\lambda_\theta^{-n}}{n}, \quad \lambda_s^{-1} T_\theta = 2\mu h \frac{\lambda_\theta^n - \lambda_s^{-n}\lambda_\theta^{-n}}{n} \tag{6.9}$$

Differentiating the above relations we obtain

$$2\mu h \lambda_s' = \frac{\lambda_s [(\lambda_\theta^n + \lambda_s^{-n}\lambda_\theta^{-n})(\lambda_\theta^{-1} T_s)' - \lambda_s^{-n}\lambda_\theta^{-n} (\lambda_s^{-1} T_\theta)']}{\lambda_s^n \lambda_\theta^n + \lambda_s^{-n} + \lambda_\theta^{-n}} \tag{6.10}$$

$$2\mu h \lambda_\theta' = \frac{\lambda_\theta [-\lambda_s^{-n}\lambda_\theta^{-n} (\lambda_\theta^{-1} T_s)' + (\lambda_s^n + \lambda_s^{-n}\lambda_\theta^{-n})(\lambda_s^{-1} T_\theta)']}{\lambda_s^n \lambda_\theta^n + \lambda_s^{-n} + \lambda_\theta^{-n}}$$

Passing now to the case of a uniaxial zone we note that for a shell of revolution (Fig.2) we have

$$ds = \sqrt{a_{11}}\, d\alpha^1 = R_1(\varphi)\, d\varphi, \quad \sqrt{a_{22}} = r = R_2 \sin \varphi \tag{6.11}$$

Taking the first of the above relations into account, we transform (5.2) to the form

$$d^2 c_1^2 (d\varphi)^2 + c_1^2 = R^{-2}$$

with the obvious general solution

$$c_1 = \sqrt{d \cos \varphi - \Delta}, \quad \Delta = e \sin \varphi + R^{-2}$$

According to (6.11) and (5.1)

$$c_2(\theta) = c = \text{const}, \quad d = 0, \quad c = 2e$$

so that, passing to T_s and T_θ (6.5) we find

$$R_1 = \frac{1}{\sqrt{\Delta}}, \quad \sqrt{a_{22}} = r = R_2 \sin \varphi = \frac{2\sqrt{\Delta}}{e} \tag{6.12}$$

$$x_3 = -\int_{\varphi_1}^{\varphi} \frac{\sin \varphi\, d\varphi}{\sqrt{\Delta}}, \quad \lambda_\theta^{-1} T_s = \frac{q}{\sqrt{\Delta}}, \quad \lambda_s^{-1} T_\theta = 0$$

7. Let us consider the shells of revolution whose deformed middle surfaces belong to the two-parameter family

$$R_1 = \frac{R_0}{(1 - \gamma \sin^2 \varphi)^{3/2}}, \quad R_2 = \frac{R_0}{(1 - \gamma \sin^2 \varphi)^{1/2}} \tag{7.1}$$

Here /10, 11/ $\gamma = 0$ corresponds to a sphere, $\gamma = -1$ to a paraboloid, $\gamma < -1$ to hyperboloids and $\gamma > -1$ to ellipsoids of revolution.

Taking into account (7.1), (6.5) and (6.6) we obtain, for $k = 0$,

$$\lambda_\theta^{-1} T_s = \mu h \bar{\sigma} \left[\frac{1}{(1 - \gamma \sin^2 \varphi)^{1/2}} \right] \tag{7.2}$$

$$\lambda_s^{-1} T_\theta = \mu h \bar{\sigma} \left[\frac{1 - \gamma \sin^2 \varphi}{(1 - \gamma \sin^2 \varphi)^{3/2}} \right], \quad \bar{\sigma} = \frac{q R_0}{2\mu h}$$

and taking into account the relation

$$' \equiv \frac{d}{ds} = \frac{\lambda_s}{R_1} \frac{d}{d\varphi} \tag{7.3}$$

which follows from (6.4), we obtain ($h = \text{const}$)

$$(\lambda_\theta^{-1} T_s)' = \tfrac{1}{2} q \lambda_s [-\gamma \sin \varphi \cos \varphi] \tag{7.4}$$

$$(\lambda_s^{-1} T_\theta)' = \tfrac{1}{2} q \lambda_s [-\gamma \sin \varphi \cos \varphi\, (3 + \gamma \sin^2 \varphi)]$$

Let us recall that the relations obtained in Sect.7 hold only for the biaxial zones when $T_s > 0,\ T_\theta > 0$, i.e. according to (7.2), when

$$\varphi < \varphi_*, \quad \sin^2 \varphi_* = \gamma^{-1} \tag{7.5}$$

This implies that condition (7.5) imposes restrictions only on the elliptical shells of revolution. If a, b are the ellipse semiaxes, then $\gamma = (a/b)^2 - 1$ and $\sin \varphi_* = [(a/b)^2 - 1]^{-1/2}$. From this it follows /1/, in particular, that when $b/a > 1/\sqrt{2} \cong 0.7$ or $\gamma < 1$, the elliptical shell is wholly in the biaxial stress state.

Computations were carried out for the potential (1.7) at $\beta = 1$. In Fig. 4, a ($\gamma = 0.5$) the design forms corresponding to curve 1 are shown by curve 2 ($n = 2$, $\bar{\sigma} = 0.89$) and curve 3 ($n = 4$, $\bar{\sigma} = 6.18$). Fig.4b shows the distribution of the quantities $\lambda_s \lambda_\theta$ and h°/h (the dashed line) corresponding to $n = 4$. In Fig.5 ($\gamma = 0.9$) the design forms corresponding to curve 1 are: curve 2 ($n = 2$, $\bar{\sigma} = 0.45$) and curve 3 ($n = 4$, $\bar{\sigma} = 0.51$). A property characteristic for sufficiently shallow shells is observed: they become shorter during inflation.

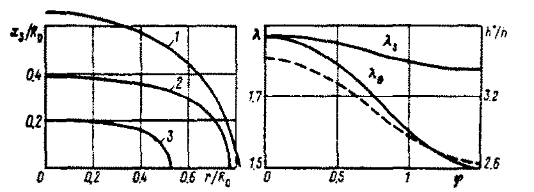

Fig.4

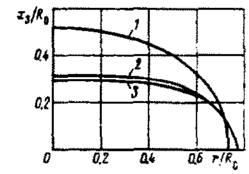

Fig.5

8. When $\gamma > 1$, only the part of the elliptical shell satisfying the condition (7.5) is in a biaxial stress state. To complement the "missing" part, we must add to it the uniaxial zone (6.12). The following obvious relations are used as conditions of coupling between the uniaxial (minus index) and biaxial (plus index) parts:

$$r^+ = r^-, \quad \varphi^+ = \varphi^-, \quad (\lambda_\theta^{-1} T_s)^+ = (\lambda_\theta^{-1} T_s)^- \tag{8.1}$$

The second of these relations ensures that there are no shearing forces which could not be balanced against anything else in the zero-moment state. When the second relation holds, the third one ensures the transfer of the vertical axial force across the line of coupling.

According to the last expression of (6.3) the natural requirement $r^{\circ+} = r^{\circ-}$ (ensuring the continuity of the design form) is equivalent to $\lambda_\theta^+ = \lambda_\theta^-$. The latter is given by the third condition of (8.1) in accordance with (6.9) $\lambda_s^+ = \lambda_s^-$ and $(\lambda_s^{-1} T_\theta)^+ = (\lambda_s^{-1} T_\theta)^-$ when $\mu^+ = \mu^-$, $h^+ = h^-$). The last equation, taking (7.2) and (6.12) into account, shows that the biaxial and uniaxial zone can be coupled to each other only when $\varphi = \varphi_*$ (7.5). Moreover, the conditions of coupling (8.1) imply that $R^{-2} = 0$, $e = 8\gamma^{1/3} R_0^{-2}$. Thus for the uniaxial zone $(S = R_0 \gamma^{-1/4} \sqrt{8})$

$$R_1 = \frac{S}{\sqrt{\sin\varphi}}, \quad r = \mathring{R}_2 \sin\varphi = 2S\sqrt{\sin\varphi}, \quad x_3 = -S\int_{\varphi_1}^{\varphi}\sqrt{\sin\varphi}\, d\varphi \tag{8.2}$$

$$\lambda_\theta^{-1} T_s = \frac{qS}{\sqrt{\sin\varphi}}, \quad \lambda_s^{-1} T_\theta = 0, \quad (\lambda_\theta^{-1} T_s)' = -\frac{q\lambda_s}{2}\operatorname{ctg}\varphi,$$

$$(\lambda_s^{-1} T_\theta)' = 0$$

The first of the relations of (7.1) and (8.2) yield $R_1^+ = R_1^-$, i.e. the curvature of the middle surface is also continuous on the matching line.

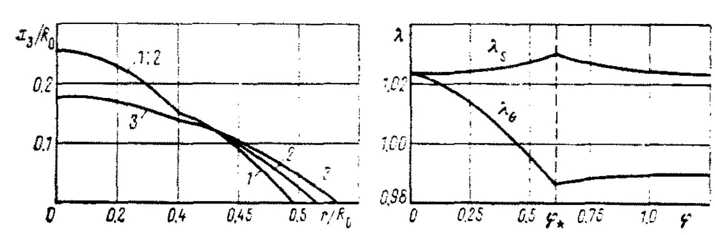

Fig.6

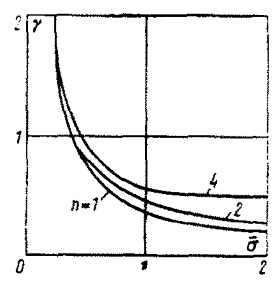

Fig.7

Thus, according to (7.2), (7.5) and (8.2).

$$(\lambda_\theta^{-1} T_s)^+ = (\lambda_\theta^{-1} T_s)^- = \mu h \bar{\sigma}/\sqrt{2}, \quad (\lambda_s^{-1} T_\theta)^+ = (\lambda_s^{-1} T_\theta)^- = 0$$

Substituting these expressions into (6.9), we obtain

$$\lambda_\theta^+ = \lambda_\theta^- = \lambda_*, \quad \lambda_s^+ = \lambda_s^- = \lambda_*^{-2}, \quad \lambda_*^{-2n} - \lambda_*^{n} = \bar{\sigma} n/(2\sqrt{2})$$

Substituting in turn the last expressions into (6.10), we obtain the following relations along the matching line:

$$2\mu h (\lambda_s')^+ = -2\mu h (\lambda_s')^- = q\sqrt{\gamma - 1}\,\frac{\lambda_*^{n-4}}{\lambda_*^{2n} + 2\lambda_*^{-n}}$$

$$2\mu h(\lambda_\theta')^+ = \frac{q\sqrt{\gamma-1}}{2}\left[-\frac{3\lambda_*^n+4\lambda_*^{-2n}}{\lambda_*(\lambda_*^{2n}+2\lambda_*^{-n})}\right].$$
$$2\mu h(\lambda_\theta')^- = \frac{q\sqrt{\gamma-1}}{2}\left[\frac{\lambda_*^n}{\lambda_*(\lambda_*^{2n}+2\lambda_*^{-n})}\right]$$

Thus the quantities $\lambda_s, \lambda_\theta$ which are continuous along the matching line, have discontinuous derivatives on this line. The relations (6.3), (6.8) yield

$$\cos\varphi^\circ = \lambda_s\lambda_\theta^{-1}\cos\varphi + r\lambda_\theta^{-2}\lambda_\theta'$$

from which we see that the discontinuity of λ_θ' implies the discontinuity of the angle on the design form.

Fig.6a shows a shell of revolution corresponding to $\gamma = 3$. Curve *1* corresponds to the surface of a two-parameter family (7.1). Curve *2* corresponds to the composite shell (with a biaxial and uniaxial zones). Fig.6b shows the discontinuities in λ_θ' and λ_s' on the matching line (the dashed line) discovered above. A family of design forms depending on the value of the parameter $\bar{s}$ corresponds to the composite shell *2*. Here the longer $\bar{s}$, the more the design form differs from the loaded form. In the course of computations a limit form (curve *3*) was discovered, corresponding to $\bar{s}_{max}$ (at large values of $\bar{s}$, the extension factors of $\lambda_s, \lambda_\theta$ no longer satisfy the inequality

$$\left|\frac{dr^c}{ds^c}\right| = \left|\lambda_s\frac{d}{ds}\left(\frac{r}{\lambda_\theta}\right)\right| \leqslant 1$$

which follows from (6.8)).

It was noted by Kolpak that $\bar{s}_{max}$ corresponds to compression of the neighbourhood of the pole of the expanded form. He also proposed that $\bar{s}_{max}$ should be determined from the compression condition $(' = \partial/\partial s^c)$

$$r_0^{c\,''} = r_0^{c\,'''} = 0, \quad \left(\cos\varphi^c(s^c) - 1 = r_0^{c\,''}s^c + \frac{1}{2}r_0^{c\,'''}s^{c2} + \ldots\right) \tag{8.3}$$

Here and henceforth the lower zero index denotes the values of the quantities when $\varphi = 0$.

Let us consider the compression condition. First, from (7.2), (7.4) and relations obtained by differentiation of the latter we obtain, taking (7.3) and (7.1) into account,

$$(\lambda_\theta^{-1}T_s)_0 = (\lambda_s^{-1}T_\theta)_0 = \mu h\bar{s}, \quad (\lambda_\theta^{-1}T_s)_0' = (\lambda_s^{-1}T_\theta)_0' = 0$$
$$(\lambda_\theta^{-1}T_s)_0'' = -\frac{q\gamma(\lambda_s)_0^2}{2R_0}, \quad (\lambda_s^{-1}T_\theta)_0'' = -\frac{3q\gamma(\lambda_s)_0^2}{2R_0}$$

Taking these equations into account we obtain, from (6.9), (6.10) and expressions obtained by their differentiation,

$$\lambda_{s0} = \lambda_{\theta 0} = \lambda_0, \quad (\lambda_0^n - \lambda_0^{-2n})/n = \tfrac{1}{2}\bar{s}; \quad \lambda_{s0}' = \lambda_{\theta 0}' = 0 \tag{8.4}$$

$$2\mu h\lambda_{s0}'' = -\frac{q\gamma\lambda_0^3}{2R}\left[\frac{\lambda_0^n - 2\lambda_0^{-2n}}{\lambda_0^{2n}+2\lambda_0^{-n}}\right] \tag{8.5}$$
$$2\mu h\lambda_{\theta 0}'' = -\frac{q\gamma\lambda_0^3}{2R_0}\left[\frac{3\lambda_0^n + 2\lambda_0^{-2n}}{\lambda_0^{2n}+2\lambda_0^{-n}}\right]$$

Taking into account the third relation of (6.3), (7.3), (7.1) and (8.4), we obtain

$$r_0 = 0, \quad r_0' = \lambda_0, \quad r_0'' = 0, \quad r_0''' = \lambda_{s0}'' - \lambda_0^3R_0^{-2} \tag{8.6}$$

Now the first relation of (6.8) and the above relations together yield

$$r_0^c = 0, \quad r_0^{c\,'} = 1, \quad r_0^{c\,''} = 0, \quad r_0^{c\,'''} = \lambda_0^{-1}[r_0''' - 3\lambda_{\theta 0}'']$$

This, together with (8.5), (8.6), implies that the first condition of (8.3) holds identically and the second condition leads to the equation

$$4\bar{s}\gamma = \frac{\lambda_0^{2n}+2\lambda_0^{-n}}{\lambda_0^n+\lambda_0^{-2n}} \tag{8.7}$$

defining, together with (8.4) $\bar{s}_{max}$, corresponding to the limiting possible design form. Fig.7 shows curves of $\bar{s}_{max}$ for various *n*.

We note that the matter discussed in Sect.8 refers to the case when the uniaxial zone is axisymmetric (without folds). The problem of the stability of such a stress-strain state requires a special consideration.

The authors thank E.P. Kolpak for valuable comments and discussions.

RERERENCES

1. ALEKSEYEV S.A., Fundamental of the theory of soft axisymmetric shells. In book: Design of Spatial Structures. 10, Moscow, Stroiizdat, 1965.
2. ALEKSEYEV S.A., Fundamentals of the theory of soft shells. In book: Design of Spatial Structures. 11, Moscow, Stroiizdat, 1967.

3. BALABUKH L.I. and USYUKIN V.I., Approximate theory of soft shells of revolution. Proceeding of the VIII-th National Conference on the Theory of Shells and Plates. Rostov-on-Don, 1971, Moscow, 1973.
4. GRIGOR'YEV A.S., Equilibrium state of a zero-moment shell of revolution under large deformations. PMM 25, 6, 1961.
5. USYUKIN V.I., On the equations of the theory of large deformations of soft shells. Izv. Akad. Nauk SSSR, MTT, 1, 1976.
6. MAGULA V.E., Elastic Ship Structures. Leningrad, Sudostroenie, 1978.
7. MAGULA V.E., Basic relationships of the theory of soft shells. Tr. Nikolaev. korablestroit in-ta, 78, 1973.
8. CHERNYKH K.F., Non-linear theory of isotropically elastic thin shells. Izv. Akad. Nauk SSSR, MTT, 2, 1980.
9. CHERNYKH K.F., Non-linear theory of thin elastic shells made of elastomers. In books: Statbility of Plates and Shells. Saratov, Izd-vo Sarat. un-ta, 1981.
10. NOVOZHILOV V.V., Theory of Thin Shells. Leningrad, Sudpromgiz, 1951.
11. CHERNYKH K.F., Linear Theory of Shells. 1, Leningrad, Izt-vo LGU, 1962.

Translated by L.K.

PMM U.S.S.R., Vol.49, No.6, pp. 763-770, 1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON THERMOELASTIC STRESSES IN AN ASYMMETRICALLY HEATED HALF-SPACE*

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A quasistatic problem of thermoelasticity is considered for a half-space in the case of convective heat exchange (boundary condition of the third kind). In the case of boundary conditions of the first and second kind all results are obtained in exactly the same manner. The exact solution of the problem is found in the form allowing the construction of an approximate solution, simple and suitable for numerical computations and based on the asymptotic expansion of the temperature and the stresses as $t \to 0$. The problem is reduced to determining single integrals of simple functions, and in many cases the integrals can be expressed in terms of elementary functions. The error of the approximate solution is estimated.

Unlike the results obtained earlier in /1–3/, the temperature distribution in the medium adjacent to the half-space is not assumed to be axisymmetric, i.e. a general asymmetric distribution is studied under certain constraints that are not significant from the physical point of view. Such asymmetric distributions are very common in practice /4/. The results of this paper can be used to study the fracture of brittle materials which can occur under the action of thermoelastic stresses /5/.

It should be noted that application of the numerical methods which were successfully used in solving the symmetric problem of thermoelasticity /6/ encounters, in the case of asymmetric, obvious difficulties caused by the increased dimensionality of the problem.

1. The initial temperature of the elastic half-space $z \geqslant 0$ and the medium filling the region $z < 0$ is $T = 0$. At the instant $t = 0$ the temperature of the medium rises instantaneously and assumes the distribution (r, φ, z are cylindrical coordinates)

$$\Theta = \Theta(r, \varphi), \quad \Theta(r, \varphi + 2\pi) = \Theta(r, \varphi) \tag{1.1}$$

and the function $\Theta(r, \varphi)$ can be written in the form of a Fourier series whose coefficients admit of the n-th order Hankel transformation in r

$$\Theta(r, \varphi) = \sum_{n=0}^{\infty} [\vartheta_n(r) \cos n\varphi + \tau_n(r) \sin n\varphi], \quad \tau_0(r) \equiv 0 \tag{1.2}$$

$$\vartheta_n^H(\lambda) = H_\lambda[\vartheta_n(r)] = \int_0^\infty r\vartheta_n(r) J_n(\lambda r)\, d\lambda$$

$$\tau_n^H(\lambda) = H_\lambda[\tau_n(r)], \quad n = 0, 1, 2, \ldots$$

*Prikl.Matem.Mekhan., 49, 6, 1001-1009, 1985